**Qualifying exam in Analysis**

**January 2015**

**You are allowed to use the statement of any part of a problem, (even if you have not solved it), if you need it in order to solve another part of the same problem. $\lambda$ denotes the Lebesgue measure, $dx$ denotes integration with respect to the Lebesgue measure and a.e. refers to the Lebesgue measure if no other measure is mentioned in the problem.**

**1)** (5 points) Let $A \subseteq \mathbb{C}$ with the following property: Every sequence $(a_n)_{n\in\mathbb{N}} \subseteq A$ such that $a_n \neq a_m$ for all $n \neq m$ converges to zero. Prove that $A$ is countable.

**2) a)** (5 points) Let $(M, d)$ be a metric space, $(x_n)_{n\in\mathbb{N}} \subseteq M$ and $x \in M$. Prove that there exists a subsequence of $(x_n)_{n\in\mathbb{N}}$ which converges to $x$ if and only if for every $\epsilon > 0$, $\sharp\{n \in \mathbb{N} : d(x_n, x) < \varepsilon\} = \infty$.

**b)** (5 points) Let $(M, d)$ be a metric space which is compact, (i.e. every open cover of $M$ has a finite subcover). Prove that every sequence $(x_n)_{n\in\mathbb{N}} \subseteq M$ has a convergent subsequence.

**3)** (10 points) Let $f : \mathcal{D} \to \mathcal{C}$ be a bounded holomorphic function where $\mathcal{D}$ denotes the unit disc. Let $d = \sup\{|f(z) - f(w)|; z, w \in \mathcal{D}\}$ denote the diameter of the range of $\mathcal{D}$ via the function $f$. Prove that $2|f'(0)| \leq d$.

**4)** (10 points) Integrate $\frac{z(z-2)}{z^3+1}$ along the circle $|z| = 3$ which is oriented counterclockwise.

**5)** (10 points) Let $f_n$ be measurable functions such that $0 \leq f_n \leq f$ a.e. and $f_n(x) \to f(x)$ a.e. Prove that $\int f_n \, dx \to \int f \, dx$. Note we do not assume that $f$ is integrable.

**6) a)** (2 points) Define what it means for a function $f : [a, b] \to \mathbb{R}$ to be absolutely continuous.

**b)** (5 points) Prove that if $f : [a, b] \to \mathbb{R}$ is absolutely continuous and $A \subseteq [a, b]$ with $\lambda(A) = 0$, then $\lambda(f(A)) = 0$.

**7)** (8 points) Let $X$ be a non-empty set, $\Sigma$ be a $\sigma$-algebra of subsets of $X$ and $f_n : X \to \mathbb{R}$ be a sequence of measurable functions which converges point-wise to a function $f : X \to \mathbb{R}$. Prove that $f$ is measurable.

**8) a)** (3 points) If $(X, \Sigma, \mu)$ is a measure space and $(A_k)_{k\in\mathbb{N}}$ is a sequence of measurable sets in $X$, prove that if $\sum_k \mu(A_k) < \infty$ then $\mu(\limsup_k A_k) = 0$.

Let $(X, \Sigma, \mu)$ be a measure space and $f_n : X \to \mathbb{R}$, $(n = 1, 2, \ldots)$, be a sequence of measurable functions. The sequence $(f_n)_{n\in\mathbb{N}}$ is called Cauchy in measure if for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\mu\{x \in X : |f_n(x) - f_m(x)| \geq \epsilon\} < \epsilon$ for all $m, n \geq n_0$. Let $(X, \Sigma, \mu)$ be a measure space and $f_n : X \to \mathbb{R}$, $(n = 1, 2, \ldots)$, be a sequence of measurable functions which is Cauchy in measure. For $\epsilon > 0$ and $n, m \in \mathbb{N}$ let

$$A^{\epsilon}_{n,m} := \{x \in X : |f_n(x) - f_m(x)| \geq \epsilon\}.$$

**b)** (2 points) Prove that there exists a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers such that $\mu(A) = 0$ where

$$A := \limsup_k A^{1/2^k}_{n_k,n_{k+1}}.$$

**c)** (3 points) Prove that $(f_{n_k}(x))_{k\in\mathbb{N}}$ is convergent for every $x \in X\backslash A$, and let $f(x)$ denote its limit.

**d)** (3 points) For every $m \in \mathbb{N}$ set

$$B_m := A \cup \bigcup_{i=m}^{\infty} A^{1/2^i}_{n_i,n_{i+1}}$$

and prove that $\mu(B_m) \to 0$.

**e)** (3 points) Prove that if $x \in X\backslash B_m$ and $i > j \geq m$ then $|f_{n_j}(x) - f_{n_i}(x)| < \frac{1}{2^{j-1}}$.

**f)** (3 points) Prove that $(f_{n_k})_{k\in\mathbb{N}}$ converges to $f$ in measure.

**g)** (3 points) Prove that $(f_n)_{n\in\mathbb{N}}$ converges to $f$ in measure.

**9)** True or False? (Prove or give counterexample as needed). Each is worth 5 points.

**a)** Every subnet of a sequence is a subsequence.

**b)** The functions $u := \sin x \cosh y$, $v := \cos x \sinh y$ satisfy the Cauchy-Riemann equations.

**c)** If $(a_{i,j})_{i,j\in\mathbb{N}}$ is an infinite matrix of real numbers such that both iterated sums $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{i,j}$ and $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{i,j}$ exist and they are real numbers, then these two iterated sums are equal.

**d)** If $f$ is a non-decreasing function and $a < b$ then $\int_a^b f'(x)dx = f(b) - f(a)$.